Lecture 3: Ideles & (maybe) Tate integrals

Recall: The ring of odeles $\mathbb{A}_K$ admits K as a discrete cocompact additive subgroup.

Eg: $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is compact

Recall
$$\mathbb{A}_{\mathbb{Q}} \subseteq \prod_{v \in V_{\mathbb{Q}}} \mathbb{Q}_v = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \ldots$$

$(x_v)_{v \in V_{\mathbb{Q}}} \in \mathbb{A}_{\mathbb{Q}} \Leftrightarrow x_v \in \mathbb{Z}_v$ for all but fin. many $v \in V_{\mathbb{Q}}$

$$x = (x_v)_{v \in V_{\mathbb{Q}}} = (x_\infty, x_2, x_3, \ldots)$$

I want to find a compact set $\Sigma \subseteq \mathbb{A}_{\mathbb{Q}}$, s.t. $\forall$ all $x \in \mathbb{A}_{\mathbb{Q}}$, $\exists y \in \mathbb{Q}$ s.t.
$$x - y \in \Sigma$$

Fix a set $\{\infty\} \subseteq S \subseteq V_{\mathbb{Q}}$ s.t.
$$x_p \in \mathbb{Q}_p \setminus \mathbb{Z}_p \Rightarrow p \in S, \quad S \text{ is a finite set}$$

eg. if $p = 2$, then
$$x_2 = \underbrace{[a_{-n}2^{-n} + a_{-n+1}2^{-n+1} + \ldots]}_{\in y} + a_0 + a_1 2 + \ldots$$

Subtracting y from x, makes $(x-y)_2 \in \mathbb{Z}_2$

To check $y \in \mathbb{Q}$ lies in $\mathbb{Z}_p$ for $p \neq 2$.

Continuing by induction, one can find $y \in \mathbb{Q}$ s.t. $x_p - y \in \mathbb{Z}_p$ for every $p \in \mathbb{P}$

But then $x_\infty - y$ is a real number, which upto addition by $z \in \mathbb{Z}$ lies in $[0,1] \subseteq \mathbb{R}$

Therefore take
$$\Sigma \subseteq \mathbb{A}_K = \underset{\mathbb{Q}_\infty}{\underset{\cap}{[0,1]}} \times (\mathbb{Z}_p)_{p \in \mathbb{P}} \leftarrow \text{compact set}$$

Ideles: $K = \#$ field, $\mathbb{A}_K$ = ring of adeles

Recall $\mathbb{A}_K$ is a ring but not a field!

Define $\mathbb{J}_K$ = the group of units in $\mathbb{A}_K$

$$\mathbb{J}_K = \bigcup_{\substack{V_K^\infty \subseteq S \subseteq V_K \\ \text{finite}}} \mathbb{J}(S)$$

$$\mathbb{J}(S) = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

Warning: $\mathbb{J}_K$ does not have the subset topology from $\mathbb{J}_K \subseteq \mathbb{A}_K$

To get topology

$$\mathbb{J}_K \to \mathbb{A}_K \times \mathbb{A}_K$$
$$x \mapsto (x, x^{-1})$$

Define: Idelic norm:

For $x = (x_v)_{v \in V_K} \in \mathbb{J}_K$

$$N_{\mathbb{J}}(x) = \prod_{\substack{v \in V_K^\infty \\ v \text{ is real}}} |x|_v \prod_{\substack{v \in V_K^\infty \\ v \text{ is complex}}} |x_v|^2 \prod_{v \in V_K^f} |x|_v$$

$N_{\mathbb{J}}: \mathbb{J}_K \to \mathbb{R}_{\geq 0}$:

Question: Why is this well-defined?

$$\boxed{\begin{array}{l} x \in K^\times \\ x \in \mathcal{O}_v^\times \text{ for all but finitely many } v \in V_K \end{array}}$$

Thm: (Artin Product formula:)

Consider $K^\times \to \mathbb{J}_K$

$$N_{\mathbb{J}}(x) = 1 \quad \text{for } x \in K^\times$$

Eg: $\mathbb{J}_{\mathbb{Q}}$, consider $1/6 \in \mathbb{Q}$

$$N_{\mathbb{J}}(1/6) = \left(\prod_{\substack{v \in V_{\mathbb{Q}}^\infty \\ v \text{ is real}}} |1/6|_v \right) \times \prod_{p \in \mathbb{P}} |1/6|_p$$

$$= 1/6 \times |1/6|_2 \times |1/6|_3 \times \cdots$$

$$= 1/6 \times \frac{|1|_2}{|6|_2} \times \frac{|1|_3}{|6|_3} \times \frac{|1|_5}{|6|_5} \times \cdots$$

$$= 1/6 \times 1/2^{-1} \times 1/3^{-1} = 1$$

One gets the following exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{J}_K^{(1)} & \longrightarrow & \mathbb{J}_K & \xrightarrow{N_{\mathbb{J}}} & \mathbb{R}_{>0} & \longrightarrow & 0 \\ & & \cup\!\! \mid & & & & & & \\ & & K^\times & & & & & & \end{array}$$

<u>Question</u>: which of the two would have infinite volume wrt Haar measure.

[Rmk: Every locally compact group has a left-invariant Haar measure]

- $\mathbb{J}_K / K^\times$ ✓ or • $\mathbb{J}_K^{(1)} / K^\times$ ✗

Justification: The map $\mathbb{J}_K \to \mathbb{R}_{>0}$ obstructs the finiteness of Haar measure

<u>Thm</u>: $\mathbb{J}_K^{(1)} / K^\times$ is compact

(every loc. compact Abelian group with finite Haar measure is compact)

Consequence: Dirichlet unit thm + finiteness of class group

Ex: Find a alg. $\mathbb{Q}$-group with no $\mathbb{Q}$-characters that proves this Thm from BHC)

"Proof of cons."

Consider $I : \mathbb{J}_K \longrightarrow \{ \mathcal{O}_K\text{-modules of } K \}$

$$(x_v)_{v \in V_K} \longmapsto \prod_{v \in V_K^f} P_v^{n_v}$$

where $|x_v|_v = N(P)^{-n_v}$

This map take an idele to a fractional ideal in K

For $x = (x_v)_{v \in V_K}$, $y \in K^\times$

$$I(xy) = y\, I(x)$$

So this gives a map

$$\frac{\left(\prod_{v \in V_K^\infty} K_v^\times\right)^{(1)} \times \left(\prod_{v \in V_K^f} \mathcal{O}_v^\times\right)}{\mathcal{O}_K^\times} \longrightarrow \frac{\mathbb{J}_K^{(1)}}{K^\times} \longrightarrow \underbrace{\frac{\{\text{frac. ideals}\}}{\sim \text{multiplication by principal ideals}}}_{\text{Class group } Cl(K)}$$

$$\frac{\left(\prod_{v \in V_K^\infty} K_v^\times\right)^{(1)} \times \left(\prod_{v \in V_K^f} \mathcal{O}_v^\times\right)}{\mathcal{O}_K^\times} \simeq \frac{(K \otimes \mathbb{R})^{(1)}}{\mathcal{O}_K^\times} \times \left(\prod_{v \in V_K^f} \mathcal{O}_v^\times\right)$$

Tate integrals:

The Riemann zeta function for $Re(s) > 1$

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Functional equation.

Put $\zeta^*(s) = \pi^{-s/2}\, \Gamma(s/2)\, \zeta_K(s)$

then $\zeta^*(s) = \zeta_K^*(1-s)$

One can write

$$\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} \cdots$$ as an integral on $\mathbb{Z}_p$

$p=2$, Want to show

$$\int_{\mathbb{Z}_p} |x|_2^s \, d^\times x = \left(1-\frac{1}{p^s}\right)^{-1}$$

where $d^\times x =$ Haar measure on $\mathbb{Q}_p^\times$

s.t. $d^\times x(\mathbb{Z}_p^\times) = 1$

Observe

$x \in \mathbb{Z}_2 \Rightarrow x = x_0 + x_1 2 + x_2 2^2 + \cdots$

$x \in \mathbb{Z}_2^\times$ iff $x_0 = 1$

$x \in 2\mathbb{Z}_2^\times$ iff $x_0 = 0 \quad x_1 = 1$

$x \in 2^2\mathbb{Z}_2^\times$ iff $x_0 = 0, \; x_1 = 0, \; x_2 = 1 \ldots$

$$\mathbb{Z}_2 = \underbrace{\mathbb{Z}_2^\times}_{\substack{vol = 1 \\ \text{wrt } d_2^\times}} \sqcup \underbrace{2\mathbb{Z}_2^\times}_{vol=1} \sqcup \underbrace{2^2\mathbb{Z}_2^\times}_{} \sqcup \cdots$$

$1 \cdot 1_2^s = 1 \qquad 1 \cdot 1_2^s = \frac{1}{2^s} \qquad 1 \cdot 1_2^s = \frac{1}{4^s}$

So $$\int_{\mathbb{Z}_2} |x|_2^s \, d^\times x = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \cdots$$

$\boxtimes$

Now $$\pi^{-s/2}\Gamma(s/2) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \frac{dx}{x}$$

Observe that $\frac{dx}{x}$ is a Haar measure $\mathbb{R}^\times$

So what we get

$\exists f : \mathbb{J}_{\mathbb{Q}}^\times \to \mathbb{R}$

Given as $f(x_\infty, x_2, x_3, \ldots) = e^{-\pi x_\infty^2} \mathbb{1}_{\mathbb{Z}_2} \cdot \mathbb{1}_{\mathbb{Z}_3} \cdots$

We get $\zeta^*(s) = \int_{\mathbb{J}_{\mathbb{Q}}^{\times}} f(x)\,|N_{\mathbb{J}}(x)|^s \, d^{\times}x$

where $d^{\times}x$ is a suitable Haar measure.

Tate integral allow one to prove analytic cont. more conseptually

Dirichet L-func., Riem. Zeta function $\longleftrightarrow \int_{\mathbb{J}_{\mathbb{Q}}}$

Hecke L-funct, Dedekind zeta function $\longleftrightarrow \int_{\mathbb{J}_K}$

Use Poisson summation

—— x —— x —— x ——