Lecture 2: Adeles & Ideles

Some references:

Lec 1	Lec 2	Lec 3
Alg. Groups & Number Th. - Platonov, Rapinchuk, Rapinchuk	Alg. Number Theory, - Neukirch	Basic Number Theory - Weil

Suppose we write in Base 2

$$(10110001)_2$$

$$(11110001)_2 \qquad (10110011)_2$$

2-adic distance on $\mathbb{Z}_{\geq 1}$

$x = x_0 + x_1 2 + x_2 2^2 + \cdots$

$y = y_0 + y_1 2 + y_2 2^2 + \cdots$

$|x-y|_2 = 2^{-n}$

where n is the first i s.t. $x_i \neq y_i$

$$|x - 0|_2 = 2^{-n}$$

where n = the first non-zero 2-adic digit of x

Observe: $|xy|_2 = |x|_2 |y|_2$ for $x, y \in \mathbb{Z}_{\geq 1}$

Extend this to $\mathbb{Q}$

$|-1|_2 = |1|_2$

and $|0|_2 = 0$

and $|x/y|_2 = |x|_2 / |y|_2$

Def: A place / abs. value on a number field K is a map $|\ |_v : K \to \mathbb{R}_{\geq 0}$

s.t. $|xy|_v = |x|_v |y|_v$, $|0|_v = 0$, $|1|_v = 1$

$|x+y|_v \leq |x|_v + |y|_v$, $|\cdot|_v \notin \{0,1\}$

Thm: (Ostrowski)

Any abs. values of K, upto equivalence is ~~#~~ one of the following

Two values are equivalent if $|\cdot|_v = |\ |_w^c$ for $c > 0$

Arch.: Take any embedding (real/complex) $\sigma : K \to \mathbb{C}$ and write $|x|_\sigma = |\sigma(x)|$

non-Arch: Take a prime ideal $P \subseteq \mathcal{O}_K$

For $x \in \mathcal{O}_K \setminus \{0\}$, $|x|_P = N(P)^{-n}$

where $n = \max\{m \in \mathbb{Z}_{\geq 0} \mid x \in P^m\}$

extend $|\cdot|_P : \mathcal{O}_K \to \mathbb{R}_{\geq 0}$ to $K \to \mathbb{R}_{\geq 0}$

To check: 1.) $K = \bigcup_{n \geq 1} \frac{1}{n} \mathcal{O}_K$

2.) $\frac{x_1}{y_1} = \frac{x_2}{y_2} \Rightarrow \frac{|x_1|_P}{|y_1|_P} = \frac{|x_2|_P}{|y_2|_P}$

$x \in K$, $\exists f \in \mathbb{Q}(x)$ st $f(x) = 0$

Local completions

Given $|\ |_v : K \to \mathbb{R}_{\geq 0}$

$$K_v = \overline{\{\text{Cauchy sequences } \{x_i\}_{i=1}^{\infty} \subseteq K\}} \Big/ \sim$$

$\{x_i\} \sim \{y_i\} \Leftrightarrow |x_i - y_i|_v \to 0$

By construction $\to K_v$ is a field

$\to K \subseteq K_v$ is dense

$$\begin{array}{ccc} K \times K & \longrightarrow & K \\ \cap\!\!| & & \downarrow \\ K_v \times K_v & \longrightarrow & K_v \\ (x, y) & \longmapsto & x + y \end{array}$$

For $\sigma : K \to \mathbb{R}$ $\quad$ $\sigma : K \to \mathbb{C}$ $\quad$ Arch.

$K_\sigma = \mathbb{R}$ $\quad$ $K_\sigma = \mathbb{C}$

For v non-Arch.

$$K_v = \{ \sum_{i \geq n}^{\infty} a_i \text{ s.t. } a_i \in p^n O_K \} \text{ for som } n \in \mathbb{Z} \}$$

Local integers

Let v be non-Arch. place

Prop:/Def:

1.) $O_v = \{x \in K_v \mid |x|_v \leq 1\}$ is a ring (ring of local integer)

2.) $O_v^\times = \{x \in K_v \mid |x|_v = 1\}$ are the units of $O_v^\times$

3.) $p \cdot O_v = \{x \in K_v \mid |x|_v < 1\}$ are an ideal inside O_v. (unique max. ideal)

"Proof": Ultrametric property

$|x+y|_v \leq \max\{|x|_v, |y|_v\} \leq |x|_v + |y|_v$

$|x x^{-1}|_v = 1$

Def:
For an ideal $I \subseteq O_K$
$I^{-1} \subseteq K$, which
$I^{-1} = \{x \in K, \ xI \subseteq O_K\}$

Observe that
$I^{-1} =$ O_K-module, f.g.

Topology of local integers

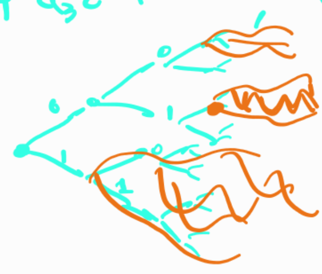
Prop: O_v is compact when v is non-Arch. wrt to $|\ |_v$-metric topology.

Eg: $\mathbb{Z}_2$ is compact

Idea: Every element in $\mathbb{Z}_2$ looks like

$a_0 + a_1 2 + a_2 2^2 + a_3 2^3 + \dots \quad a_i \in \{0, 1\}$

Every open cover has a finite subcover?



Adeles:

Let $V_K =$ all places of K

$V_K^\infty =$ Arch.

$V_K^f =$ non-Arch.

For a finite set $V_K^\infty \subseteq S \subseteq V_K$

denote $\mathbb{A}_K(S) = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$

$$\mathbb{A}_K = \bigcup_{\substack{V_K^\infty \subseteq S \subseteq V_K \\ \text{finit.}}} \mathbb{A}_K(S)$$

Observe that each $\mathbb{A}_K(S)$ is loc. compact

$\Rightarrow \mathbb{A}_K$ is a loc. compact topological ring.

($\because$ Each $\mathbb{A}_K(S)$ is a ring)

Question: Is $\mathbb{A}_K$ a field?

Take $(x_v)_{v \in V_K}$ s.t.

$x_p \in P$ $\forall$ each prime ideal $P \subseteq \mathcal{O}_K$

Diagonal embedding of $K \hookrightarrow \mathbb{A}_K$:

$K \to \mathbb{A}_K$

$x \longmapsto (x_v)_{v \in V_K}$ $\quad x_v \in K_v \supseteq K$, $x_v = x$

To check, that except finitely many $v \in V_K$ $x_v \in \mathcal{O}_v$.

(Hint: $\exists n \in \mathbb{Z}_{\geq 1}$ s.t. $nx \in \mathcal{O}_K$

Then $\underset{x}{\underset{\shortparallel}{(x_v)_{v \in V_K}}} \in \mathbb{A}_K(S)$

where S has primes dividing n)

Thm: $K \to \mathbb{A}_K$ maps K as a discrete cocompact subgroup (i.e. $\mathbb{A}_K/K$ is compact)

"Proof": 1.) Find a nbhd $0 \in U \subseteq \mathbb{A}_K$ s.t. $U \cap K = \{0\}$

2.) Find a compact set $\Sigma \subseteq \mathbb{A}_K$ s.t. $K + \Sigma = \mathbb{A}_K$

Lemma:

1.) For $x \in K$, if for each $v \in V_K^f$ $|x|_v \leq 1$ $\Rightarrow$ $x \in \mathcal{O}_K$

Hint: $\mathbb{A}_K$ carries the smallest topology st. $\mathbb{A}_K(S)$ are open subrings.

tychonoff topology

2.) $\mathcal{O}_K \subseteq \prod_{v \in V_K^\infty} K_v \cong K \otimes \mathbb{R}$ embedded diagonally is a discrete cocompact subgroup.

$U \subseteq \mathbb{A}_K(S)$ $S = V_K^\infty$